LOCAL PRESENTABILITY OF (CO)MONADS IN DOUBLE CATEGORIES

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SECTION 0

Motivation

Proposition [Porst]

If $\mathcal{V}$ is a *locally presentable*, symmetric monoidal category whose *tensor product is finitary in each variable* then

1. $\mathrm{Mon}(\mathcal{V})$ is finitary monadic over $\mathcal{V}$ and locally presentable;
2. $\mathrm{Comon}(\mathcal{V})$ is comonadic over $\mathcal{V}$ and locally presentable.

Question: Under what conditions can we generalize these results to the context of double categories?

$$\text{monoidal } \mathcal{V} \quad \equiv \quad \begin{matrix}\text{1-object} \\ \text{bicategory}\end{matrix} \quad \xrightarrow{\text{generalize to}} \quad \text{double category } \mathbb{D}$$

$$\text{(co)monoids in } \mathcal{V} \quad \xrightarrow[\text{pass to}]{} \quad \text{(co)monads in } \mathbb{D}$$

SECTION 1

Locally presentable double categories

Many examples where the same objects can have two kinds of arrows which usually coexist and interact. Double categories provide a 2-dimensional categorical framework for such contexts.

Definition

A ***(pseudo) double category*** is a (pseudo) category object inside Cat.

So, a double category $\mathbb{D}$ consists of a *category* $\mathbb{D}_0$ of objects and a *category* $\mathbb{D}_1$ of arrows, and source, target, composition and identity *functors*

$$\mathbb{D}_0 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\underset{\longleftarrow}{\overset{\longleftarrow}{1 \longrightarrow}}}} \mathbb{D}_1 \xleftarrow{\odot} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1,$$

Subject to the usual axioms and *coherence conditions*.

Takeaway: A double category $\mathbb{D}$ consists of:

- *objects* $X, Y, Z, \ldots$;
- *tight arrows* which we denote as $X \to Y$;
- *loose arrows* which we denote as $X \nrightarrow Y$; and
- *cells* of the form

$$\begin{array}{ccc} X & \overset{P}{\nrightarrow} & Y \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ Z & \underset{Q}{\nrightarrow} & W \end{array}$$

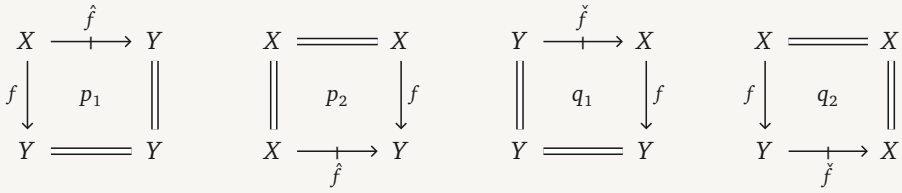
In many important examples the tight and the loose arrows are more closely related.

Definition [Shulman]

A pseudo double category $\mathbb{D}$ is called ***fibrant*** if $\langle \mathfrak{s}, \mathfrak{t} \rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a (bi)fibration.

In a fibrant double category, we can *turn tight arrows to loose arrows* thus obtaining a rich *interplay between the categories $\mathbb{D}_0$ and $\mathbb{D}_1$*.

Takeaway: For any tight arrow $f : X \to Y$ we get two loose arrows, its ***companion*** $\hat{f} : X \nrightarrow Y$ and its ***conjoint*** $\check{f} : Y \nrightarrow X$ together with cells



subject to certain conditions.

Example: the double category of (co)spans

Suppose $\mathscr{C}$ is a category *with pullbacks*.

The double category $\mathbb{S}\mathrm{pan}_{\mathscr{C}}$ of spans in $\mathscr{C}$ consists of *objects*, *arrows*, *spans*, and *span homomorphisms in* $\mathscr{C}$. A typical cell here is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{P} & Z \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[Q]{} & Z' \end{array} \quad \equiv \quad \begin{array}{ccccc} X & \longleftarrow & Y & \longrightarrow & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle g} \\ X' & \longleftarrow & Y' & \longrightarrow & Z' \end{array} \quad \text{in } \mathscr{C}.$$

Composition of spans is given by *pullbacks*, the identities are the evident ones.

$\mathbb{S}\mathrm{pan}_{\mathscr{C}}$ is *fibrant*: any arrow $f : X \to Y$ induces the *companion* span

$$X \;=\!=\; X \xrightarrow{f} Y$$

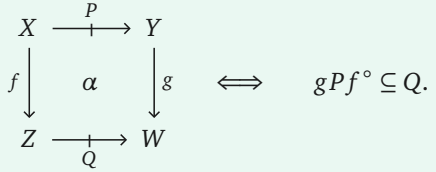
and the *conjoint* span

$$Y \xleftarrow{f} X \;=\!=\; X.$$

Dually, we can form the double category $\mathrm{co}\mathbb{S}\mathrm{pan}_{\mathscr{C}}$ of cospans in $\mathscr{C}$.

Example: the double category of relations

If $\mathscr{C}$ is also *regular*, we can form the double category $\mathbb{R}\mathrm{el}_{\mathscr{C}}$ which consists of *objects*, *arrows*, *relations*, and *inclusions of relations in* $\mathscr{C}$:



$$\iff \quad gPf^\circ \subseteq Q.$$

Compositions and identities are the evident ones.

$\mathbb{R}\mathrm{el}_{\mathscr{C}}$ is *fibrant*: any arrow $f : X \to Y$ induces two relations, its *graph* (companion) which is f viewed as a relation, and its *opposite graph* (conjoint) f°.

Example: $\mathcal{V}$-matrices

Suppose $\mathcal{V}$ is a *monoidal* category *with small coproducts preserved by* $\otimes$ *in each variable*.

The double category $\mathcal{V}$-$\mathbb{M}$at consists of *sets*, *functions*, *families* $\{P_{y,x}\}_{(y,x)\in Y\times X}$ *of objects of* $\mathcal{V}$ with X, Y sets called $\mathcal{V}$-**matrices** and *morphisms between families*. Which means that a cell is a family of functions

$$\begin{array}{ccc} X & \xrightarrow{\ P\ } & Y \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[\ Q\ ]{} & W \end{array} \quad \Longleftrightarrow \quad \{\alpha_{y,x} : P_{y,x} \to Q_{g(y),f(x)}\}_{(y,x)\in Y\times X}$$

Composition of two $\mathcal{V}$-matrices $P : X \nrightarrow Y$ and $Q : Y \nrightarrow Z$ is given by the family $\left\{\sum_{y\in Y} Q_{z,y} \otimes P_{y,x}\right\}$.

$\mathcal{V}$-$\mathbb{M}$at is *fibrant*: for any function $f : X \to Y$, its companion and conjoint are given by the same formula,

$$\{I \text{ if } f(x) = y \text{ and } 0 \text{ otherwise.}\}$$

Definition [Aravantinos-Sotiropoulos, Vasilakopoulou]

A *fibrant* double category $\mathbb{D}$ is called **locally presentable** if:

- $\mathbb{D}_0, \mathbb{D}_1$ are *locally presentable*,
- $\mathfrak{s}, \mathfrak{t}$ are *cocontinuous right adjoints*, and
- for every object A of $\mathbb{D}_1$, *the functors* $- \odot A$ *and* $A \odot -$ *are accessible*.

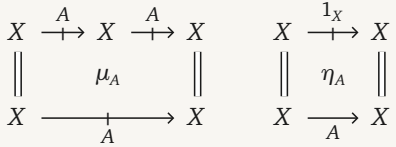
Examples [Aravantinos-Sotiropoulos, Vasilakopoulou]

1. $\mathbb{S}\mathrm{pan}_{\mathscr{C}}$ is locally presentable as long as $\mathscr{C}$ is.
2. $\mathbb{R}\mathrm{el}_{\mathsf{Set}}$ is locally presentable.
3. $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ is locally presentable if $\mathcal{V}$ is and $\otimes$ is accessible in each variable.

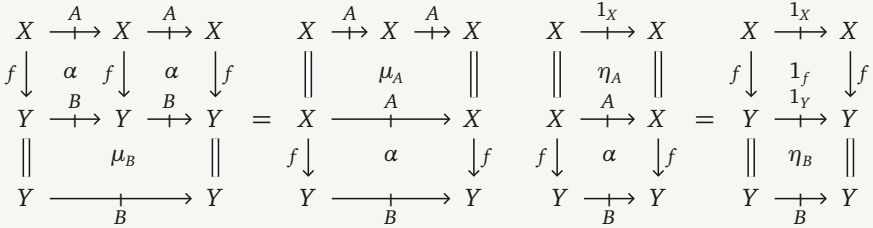
SECTION 2

(co)Monads in double categories

A **monad** in a double category $\mathbb{D}$ is some $A_X : X \nrightarrow X$ equipped with *multiplication* and *unit* cells



satisfying the usual associativity and unit laws. Given two monads A_X, B_Y, a **monad morphism** is a cell $\alpha : A_X \Rightarrow B_Y$ that *respects the multiplication and the units*:



Monads and monad morphisms assemble into a category, $\mathrm{Mnd}(\mathbb{D})$. Dually, we get **comonads** and **comonad morphisms** which assemble into the category $\mathrm{Cmd}(\mathbb{D})$.

Examples

1 $\mathrm{Mnd}(\mathbb{S}\mathrm{pan}_{\mathscr{C}}) = \mathrm{Cat}_{\mathscr{C}}$:

loose endo-arrow $A_{X_0} : X_0 \nrightarrow X_0$	gives	source and target $X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0$
multiplication $\mu_A : A \odot A \Rightarrow A$		composition $X_1 \times_{X_0} X_1 \to X_1$
unit $\eta_A : 1_A \Rightarrow A$		identity $X_0 \to X_1$
monad morphism		internal functor

Comonads in $\mathbb{S}\mathrm{pan}_{\mathscr{C}}$ are just arrows; the counit implies equality of the legs.

2 $\mathrm{Mnd}(\mathbb{R}\mathrm{el}_{\mathscr{C}}) = \mathrm{Preord}_{\mathscr{C}}$:

loose endo-arrow $R_X : X \nrightarrow X$	gives	relation $R \subseteq X \times X$
multiplication $\mu_R : R \odot R \Rightarrow R$		$RR \subseteq R \iff R$ is transitive
unit $\eta_R : 1_R \Rightarrow R$		$\Delta_X \subseteq R \iff R$ is reflexive
monad morphism		order preserving morphism

A comonad $C_X : X \to X \in \mathrm{Cmd}(\mathbb{R}\mathrm{el}_{\mathscr{C}})$ is just a subobject of X; the counit implies $C \subseteq \Delta_X$.

3 $\mathrm{Mnd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathrm{Cat}$:

loose endo-arrow $A_X : X \nrightarrow X$	gives	family of objects $A_{x,x}$ in $\mathcal{V}$
multiplication $\mu_A : A \odot A \Rightarrow A$		morphisms $A_{x',x''} \otimes A_{x,x'} \to A_{x,x''}$
unit $\eta_A : 1_A \Rightarrow A$		$I \to A_{x,x'}$
monad morphism		$\mathcal{V}$-functors

Dually, $\mathrm{Cmd}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at}) = \mathcal{V}\text{-}\mathrm{coCat}$.

SECTION 3

Local presentability of $\mathsf{Mnd}(\mathbb{D})$ and $\mathsf{Cmd}(\mathbb{D})$

There are obvious forgetful functors

$$\begin{array}{rcl} \mathsf{Mnd}(\mathbb{D}) & \to & \mathsf{End}(\mathbb{D}) \\ (X \overset{A_X}{\nrightarrow} X, \mu_A, \eta_A) & \mapsto & X \overset{A_X}{\nrightarrow} X \end{array}$$

and

$$\begin{array}{rcl} \mathsf{Cmd}(\mathbb{D}) & \to & \mathsf{End}(\mathbb{D}) \\ (Y \overset{C_Y}{\nrightarrow} Y, \delta_C, \varepsilon_C) & \mapsto & Y \overset{C_Y}{\nrightarrow} Y \end{array}$$

Then, we proved that *under certain assumptions on* $\mathbb{D}$,

- there is a (co)free (co)monad,
- (co)monads are (co)monadic over $\mathsf{End}(\mathbb{D})$ as soon as the free (co)monad exists, and
- *the categories* $\mathsf{Mnd}(\mathbb{D})$ *and* $\mathsf{Cmd}(\mathbb{D})$ *are locally presentable*.

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

If $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits then $\mathsf{Mnd}(\mathbb{D})$ *is locally presentable.*

Proof. Consider the F-algebras for the endofunctor

$$F : \mathsf{End}(\mathbb{D}) \to \mathsf{End}(\mathbb{D})$$
$$A_X \mapsto (A_X \odot A_X) + 1_X,$$

that is some $A_X : X \nrightarrow X$ in $\mathsf{End}(\mathbb{D})$ equipped with a structure map

$$\alpha : FA_X \Rightarrow A_X \iff \alpha : (A_X \odot A_X) + 1_X \Rightarrow A_X \iff \begin{cases} \alpha_1 : A_X \odot A_X \Rightarrow A_X \\ \alpha_2 : 1_X \Rightarrow A_X \end{cases}$$

F is finitary
$\mathsf{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\Big\}$ $\Rightarrow \mathsf{Alg}(F)$ is locally presentable.

$\mathsf{Mnd}(\mathbb{D})$ is a full subcategory of $\mathsf{Alg}(F)$ closed under limits and filtered colimits. Thus, by the Reflection Theorem, $\mathsf{Mnd}(\mathbb{D})$ is also locally presentable. □

The above theorem *cannot* be dualized.

Theorem (Aravantinos-Sotiropoulos, T., Vasilakopoulou)

Suppose $\mathbb{D}$ is locally presentable and the functors $- \odot -$ and $1_{(-)}$ preserve filtered colimits. If furthermore, $\mathbb{D}$ has stable local initial objects, then $\mathsf{Cmd}(\mathbb{D})$ is locally presentable.

Proof. Consider the F-coalgebras for the endofunctor

$$F : \mathsf{End}(\mathbb{D}) \to \mathsf{End}(\mathbb{D})$$
$$C_X \mapsto (C_X \odot C_X) \times 1_X,$$

that is some $C_X : X \nrightarrow X$ in $\mathsf{End}(\mathbb{D})$ equipped with a structure map

$$\gamma : C_X \Rightarrow F C_X \iff \gamma : C_X \Rightarrow (C_X \odot C_X) \times 1_X \iff \begin{cases} \gamma_1 : C_X \Rightarrow C_X \odot C_X \\ \gamma_2 : C_X \Rightarrow 1_X \end{cases}$$

F is finitary
$\mathsf{End}(\mathbb{D})$ is locally presentable since $\mathbb{D}$ is $\Big\}$ $\Rightarrow \mathsf{Coalg}(F)$ is locally presentable.

$\mathsf{Cmd}(\mathbb{D})$ is contained in $\mathsf{Coalg}(F)$ but for F-coalgebras, their structure maps are neither globular nor do they need to satisfy any coassociativity and counitality conditions.

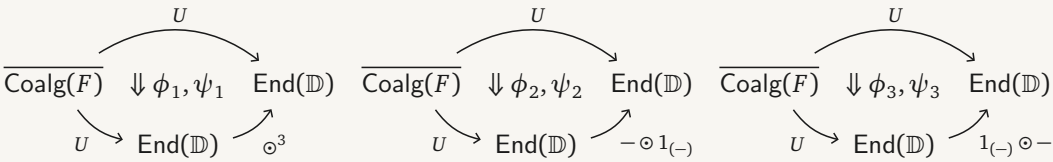
First we *fix the globularity of the structure cells*.

Consider the full subcategory $\overline{\mathsf{Coalg}(F)}$ of $\mathsf{Coalg}(F)$ spanned by F-coalgebras with *globular* structure maps.

$\left.\begin{array}{l}\overline{\mathsf{Coalg}(F)} \text{ is closed under all colimits, thus cocomplete} \\ \overline{\mathsf{Coalg}(F)} \text{ is closed under all subobjects } \textbf{(!)} \text{ hence accessible}\end{array}\right\} \Rightarrow \overline{\mathsf{Coalg}(F)} \text{ is locally presentable.}$

Now we *force coassociativity and counitality axioms*.

$\mathsf{Cmd}(\mathbb{D})$ can be described as an equifier



All functors are accessible, therefore $\mathsf{Cmd}(\mathbb{D})$ is accessible and cocomplete, hence locally presentable. □

REFERENCES

- J. Adámek, J. Rosický, Locally presentable and Accessible Categories, London Mathematical Society Lecture Note Series 189, Cambridge University Press.
- V. Aravantinos-Sotiropoulos, T. Tsantilas, C. Vasilakopoulou, On categories of monads and comonads in double categories, arXiv:2606.26755.
- V. Aravantinos-Sotiropoulos, C. Vasilakopoulou, Sweedler theory for double categories, arXiv:2408.03180.
- H.-E. Porst, On categories of monoids, comonoids, and bimonoids, Quaest. Math., 31(2):127–139, 2008.
- M. Shulman, Framed bicategories and monoidal fibrations, Theory and Applications of Categories, 20(18):650–738, 2008.

Thank you for your attention!